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LETTER TO THE EDITOR

On the convergence of a class of random geometric series with application to random walks and percolation theory

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Abstract. We consider a class of random geometric series with an underlying tree-like structure that has a number of applications in statistical physics. Convergence criteria for these series are discussed and consistency with different criteria known to hold in the one-dimensional limit is established. A multiplicative, two-component model of percolation on a Cayley tree is defined and analysed. The order of the percolation transition and certain critical exponents are altered compared to conventional bond percolation.

Consider a (quenched) set of positive, random variables $\{X_j\}$ that are independent and identically distributed (iid) and generated by a probability density $\rho(x)$. The following infinite, random geometric series arises naturally in a number of one-dimensional problems encountered in statistical physics and, as such, has been widely studied [1–4],

$$R = X_1 + X_1X_2 + X_1X_2X_3 + \cdots. \tag{1}$$

The following theorem concerning the random variable R is given by Kesten [5–7].

Theorem 1. (i) The sequence R (equation (1)) converges with probability one if and only if $\langle \ln x\rangle < 0$, when R is a random variable with a well defined distribution. (ii) The sequence R (equation (1)) diverges with probability one if and only if $\langle \ln x\rangle > 0$. The average is defined with respect to the density $\rho(x)$.

Equation (1) is interesting because the multiplicative and correlated nature of the series leads to critical behaviour representative of a phase transition. To put this into a more general context, many physical processes have been studied in terms of sums of random variables [7]:

$$S_N = \sum_{i=1}^{N} Y_i. \tag{2}$$

For example, if $\{Y_i\}$ is a set of iid variables generated by a probability density $\phi(y)$, then we have a random-walk model of diffusion (the laws of large numbers). Assuming that $\phi(y)$ decays for large arguments as $y^{-1-\mu}$ then, if $\mu > 2$, the limiting distribution for S_N (when suitably scaled) is Gaussian (central limit theorem) and the diffusion is termed conventional. If, on the other hand, $\phi(y)$ is sufficiently broad ($0 < \mu < 2$), the limiting distribution for S_N (a Levy stable law of index μ) is less generic, in that the summation tends to be dominated by its largest term, leading to anomalous diffusion [7]. Less well understood is what happens when the variables $\{Y_i\}$ are correlated in some way, as in equation (1). Weak correlations lead to an effective renormalization of the above picture, but strong correlations can lead to qualitatively different (even critical) behaviour. This is manifest in the convergence or otherwise of equation (1). A biased random walk in one dimension is persistent or transient, when the bias field is spatially random, according to the convergence or otherwise of a series identical to equation (1) [1, 2, 5, 6]. Another problem which maps onto equation (1) is that of the typical escape time for a particle in a trapping region undergoing a succession of thermally activated jumps [7]. Individual terms in equation (1) have also been studied as multiplicative cascade models of turbulence, and give rise to random measures which are multifractals [8].

Motivated by statistical physics, the aim of this letter is to discuss the generalization of equation (1) from one dimension to an arbitrary tree, in the following sense. Consider an infinite tree Γ (see figure 1). With each branch j of the tree associate an independent and identically distributed (iid) random variable $X_j \geqslant 0$ generated by a probability density $\rho(x)$. Once generated, the set $\{X_j\}$ is fixed. For a given branch i, construct the unique shortest path joining it to the origin. Using the values of $\{X_j\}$ for all the branches on the path, construct the following random product which is specific to branch i,

$$R_i = X_1 X_2 X_3 \ldots X_i. \tag{3}$$

Now form the sum of the R_i over all the branches of Γ, which will generate an infinite, random geometric series,

$$R = \sum_{\Gamma} R_i. \tag{4}$$

The one-dimensional limit (equation (1)) corresponds to the special case where the tree is regular with branching number z and $z = 1$. The question as to the convergence or otherwise of equation (4) has been discussed by Lyons and co-workers in the context of the persistence or recurrence of random walks on trees [9, 10], and by the present authors in relation to demonstrating a localization–delocalization transition for drift diffusion in a quenched random velocity field [11]. Below we present the relevant theorem (which we suspect is not widely known) in its most general setting. The proof, based on network flow theory, is long and technical and is omitted.

For an arbitrary tree one can define the *branching number* $\mathrm{br}(\Gamma)$ according to

$$\mathrm{br}(\Gamma) = \inf\left\{\lambda > 0; \inf_{\Pi} \sum_{i\in\Pi} \lambda^{-|i|} = 0\right\}$$

where Π is a cutset, i.e. a finite set of vertices excluding the origin such that every path from the origin to infinity intersects Π and such that there is no pair $i, j \in \Pi$ with $i < j$. A special example of a cutset is the nth generation S_n, $n \geqslant 1$. Here $\mathrm{br}(\Gamma)$ is a measure of the average number of branches per vertex of Γ. It is less than or equal to the so-called *growth rate*

$$\mathrm{gr}(\Gamma) = \lim_{n\to\infty} \inf \Sigma_n^{1/n}$$

where Σ_n is the number of branches in the nth generation (see figure 1). For a regular tree of branching number z, $\mathrm{br}(\Gamma) = \mathrm{gr}(\Gamma) = z$. Define the function $\beta(\sigma)$ as follows (where $\sigma \geqslant 0$):

$$\beta(\sigma) = \langle x^\sigma \rangle = \int_0^\infty \rho(x) x^\sigma \,\mathrm{d}x. \tag{5}$$

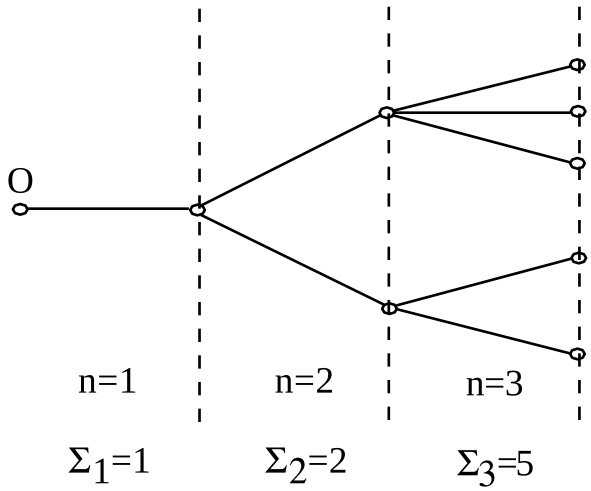

Figure 1. A locally finite tree Γ indicating successive generations defined with respect to the origin O.

Introduce the index σ^* ($0 \leqslant \sigma^* \leqslant 1$) according to the property

$$\beta(\sigma^*) = \min_{0\leqslant\sigma\leqslant 1} \beta(\sigma). \tag{6}$$

Note that σ^* depends only upon the probability density $\rho(x)$. Assume (initially) that $\rho(x)$ is such that X is positive definite with probability one. We have

Theorem 2. (i) The sequence R (equation (4)) converges with probability one if and only if $\mathrm{gr}(\Gamma)\beta(\sigma^*) < 1$, when R is a random variable with a well defined distribution. (ii) The sequence R (equation (4)) diverges with probability one if and only if $\mathrm{br}(\Gamma)\beta(\sigma^*) > 1$.

If $\rho(x)$ is such that X is zero with positive probability, then theorem 2 still holds provided the statement 'with probability one' in case (ii) is replaced by the statement 'with positive probability'. This subtle distinction will be important in what follows.

We now demonstrate that theorem 1 and theorem 2 are equivalent (or at least consistent) in one dimension. This is not immediately obvious and, to the best of our knowledge, has not been discussed before. First, note from equation (5) that $\beta(\sigma)$ is a convex function,

$$\frac{\partial^2\beta(\sigma)}{\partial\sigma^2} = \int_0^\infty \rho(x)(\ln x)^2 x^\sigma\,\mathrm{d}x > 0. \tag{7}$$

Second, note that $\beta(0) = 1$. Since $\beta(\sigma)$ is convex, it follows that if $\partial\beta(0)/\partial\sigma \geqslant 0$, then $\sigma^* = 0$ and $\beta(\sigma^*) = 1$. If, on the other hand, $\partial\beta(0)/\partial\sigma < 0$, then there must be a value of $\sigma > 0$ for which $\beta(\sigma) < 1$; i.e. $\sigma^* > 0$ and $\beta(\sigma^*) < 1$. From equation (5) we obtain

$$\frac{\partial\beta(0)}{\partial\sigma} = \int_0^\infty \rho(x)\ln x\,\mathrm{d}x = \langle\ln x\rangle. \tag{8}$$

Thus if $\langle\ln x\rangle < 0$, then $\beta(\sigma^*) < 1$ and the series is convergent with probability one according to both theorems. If $\langle\ln x\rangle > 0$ then, according to theorem 1, the series is divergent with probability one. According to theorem 2, $\beta(\sigma^*) = 1$ and the series diverges with probability one for any $z = 1 + \epsilon$ for which $\epsilon > 0$. If $z = 1$ then theorem 2 is indeterminate. Thus theorem 2 is consistent with theorem 1 in one dimension but theorem 1 is stronger. However, theorem 2 is more powerful in the sense that it is valid for any $z \geqslant 1$. Note that theorem 1 is only indeterminate when $\langle \ln x \rangle = 0$, when theorem 2 is also indeterminate (for $z = 1$) since $\partial\beta(0)/\partial\sigma = 0$, $\sigma^* = 0$ and $\beta(\sigma^*) = 1$.

A simple example helps to illustrate the use of theorem 2. Consider a regular tree of branching number z and consider the case where X is uniformly distributed on the interval $[0, L]$. Then,

$$\beta(\sigma) = \frac{1}{L}\int_0^L x^\sigma \, \mathrm{d}x = \frac{L^\sigma}{\sigma + 1}.$$

There are three cases of interest.

(i) $0 < L \leqslant \sqrt{e}$, for which $\sigma^* = 1$ and $\beta(\sigma^*) = L/2 < 1$. The series is convergent with probability one if $z < 2/L$ and divergent with probability one if $z > 2/L$. For $z = 1$, the series is convergent with probability one.

(ii) $\sqrt{e} < L < e$, for which $\sigma^* = (1/\ln L - 1)$ and $\beta(\sigma^*) = e \ln L/L < 1$. The series is convergent with probability one if $z < L/(e \ln L)$ and divergent with probability one if $z > L/(e \ln L)$. For $z = 1$, the series is convergent with probability one.

(iii) $L \geqslant e$, for which $\sigma^* = 0$ and $\beta(\sigma^*) = 1$. The series is divergent with probability one for all $z > 1$. For $z = 1$, theorem 2 is indeterminate.

It is readily checked that the conclusions reached for $z = 1$ are consistent with those reached using theorem 1. In addition, theorem 1 tells us that the series is divergent with probability one for $L > e$.

We shall now consider a more interesting example; a choice for $\rho(x)$ that maps equation (4) onto the problem of conventional bond percolation on a tree. Consider a regular Cayley tree with $z > 1$ (the case $z = 1$ is slightly different but can be handled similarly). It will prove useful to consider the Laplace transform of the probability density $\Psi(r)$ of the random variable R (assuming it exists),

$$M(s) = \int_0^\infty \mathrm{e}^{-sr}\Psi(r)\,\mathrm{d}r \qquad M(0) = 1. \tag{9}$$

For a regular tree, the well defined recursive structure of the problem means that $M(s)$ obeys the following integral equation [11],

$$M(s) = \int_0^\infty \rho(x)\mathrm{e}^{-sx}[M(sx)]^z \,\mathrm{d}x. \tag{10}$$

Consider now the density $\rho(x) = (1-p)\delta(x-\epsilon) + p\delta(x-1)$, $0 \leqslant p \leqslant 1$, $\epsilon < 1$, for which $\sigma^* = 1$, $\beta(\sigma^*) = \epsilon(1-p) + p$, and equation (10) reduces to the following form,

$$M(s) = (1-p)\mathrm{e}^{-s\epsilon}M(s\epsilon)^z + p\mathrm{e}^{-s}M(s)^z. \tag{11}$$

Two values of X are permitted, $X = \epsilon$ (with probability $1-p$) and $X = 1$ (with probability p). Suppose that ϵ is chosen to be strictly zero ($\beta(\sigma^*) = p$). Those branches of Γ that (i) have a value of $X = 1$ and (ii) are connected to the origin by an unbroken sequence of branches all with $X = 1$ have a value of $R_i = 1$; all other branches have $R_i = 0$. Thus R (equation (4)) represents the size (or mass) of the cluster connected to the origin. This is the classic problem of bond percolation on a Cayley tree, and its well known behaviour [8] can be rederived quickly and elegantly from a new perspective. Theorem 2 tells us immediately that R is finite with probability one (no infinite cluster exists) if $p < 1/z$. If $p > 1/z$ then R is infinite with positive probability, since X is zero with positive probability (this means an infinite cluster exists for certain realizations, but not for others). This behaviour is indicative of a second-order (continuous) phase transition at $p = p_\mathrm{c} = 1/z$, where the probability that the origin is linked to an infinite cluster is the order parameter. One can show from equation (11) that the moments of $\Psi(r)$ diverge as one approaches the percolation threshold from below as

$$M_n \equiv \int_0^\infty r^n \Psi(r)\,\mathrm{d}r \equiv (-1)^n \frac{\mathrm{d}^n M(0)}{\mathrm{d}s^n} \sim (p_\mathrm{c} - p)^{-2n+1}. \tag{12}$$

Using arguments from scaling theory it follows that the asymptotic form for $\Psi(r)$ can be written as

$$\Psi(r) \sim r^{-\tau} F(r^\sigma (p_\mathrm{c} - p)) \tag{13}$$

where F is some function and the critical exponents are given by $\sigma = 1/2$ and $\tau = 3/2$ [8]. When $p = p_\mathrm{c}$ the density $\Psi(r)$ still exits (is normalizable). For completeness, the exponent which governs the divergence of the first moment is given by $\gamma = 1$.

Mathematically one can ask what happens in the above example when ϵ is strictly positive? Every branch on the tree now contributes to equation (4). The threshold for the convergence of equation (4) shifts such that

$$p_\mathrm{c} = \frac{1}{z}\frac{1 - z\epsilon}{1 - \epsilon} \qquad 0 < \epsilon < z^{-1} \tag{14}$$

and $p_\mathrm{c} = 0$ for $\epsilon \geqslant z^{-1}$. For $p < p_\mathrm{c}$ the mass of the cluster (R) is finite with probability one, as before. However, for $p > p_\mathrm{c}$ the mass of the cluster is infinite with probability one, not just with positive probability as before, since X is now positive definite with probability one. This means the phase transition at $p = p_\mathrm{c}$ is now first order (discontinuous) rather than second order. When $p < p_\mathrm{c}$ we can still define the same exponents as above. From equation (11) one deduces, for $p_\mathrm{c} - p \ll \epsilon$,

$$M_n \sim (p_\mathrm{c} - p)^{-n}. \tag{15}$$

The mean cluster mass still diverges at the critical point with the same critical exponent $\gamma' = 1$, but the other exponents are different. For large arguments,

$$\Psi(r) \sim r^{-\tau'} F'(r^{\sigma'} (p_c - p)) \tag{16}$$

where $\sigma' = 1$ and $\tau' = 1$. When $p = p_\mathrm{c}$ the density $\Psi(r)$ no longer exists (is not normalizable). This is a consequence of the transition being first order rather than second order. The scaling form will no longer be valid if $p_\mathrm{c} - p \sim \epsilon$.

One can interpret the above choice for $\rho(x)$ as *defining* a generalized two-component, multiplicative percolation process. Does such a process have any direct physical interpretation? One example, an idealization of a number of physical problems (see e.g. [11]), is as follows. Suppose one fixes the concentration of some species at the root of an infinite tree and invasion percolation along the tree's branches takes place via drift diffusion. The (dimensionless) drift velocities (towards the origin) on each branch are randomly chosen to be either zero (with probability p) or $\ln 1/\epsilon$ (with probability $1 - p$). Those links with velocity $\ln 1/\epsilon$ act as bottlenecks to the invasion process; when $\epsilon = 0$ these bottlenecks are impenetrable. A relevant question is: under what conditions (once steady state has been attained) will the total mass of invading species be infinite? This is an extension to the conventional percolation idea of connecting to infinity. The problem can be mapped (essentially) onto a series like equation (4) [11]. If $p < p_\mathrm{c}$ the invading mass will be finite for any realization. If $p > p_\mathrm{c}$ then, provided $\epsilon > 0$, the invading mass will be infinite for any realization (first-order transition). However, if $\epsilon = 0$, the invading mass will sometimes be infinite and sometimes be finite (second-order transition), depending upon the particular realization. The essential point is that impenetrable bottlenecks fundamentally affect the invasion process (transition order, exponents etc) by altering the connectivity of the system.

Alternative choices for $\rho(x)$ and generalizations to irregular trees can also be studied within the above framework.

The above analysis applies only to tree structures; that is, graphs without loops. Attempts to generalize to finite-dimensional, regular lattices meet with formidable mathematical problems. For example, the above drift-diffusion process on a square lattice with random velocities on each bond cannot be cast into the form of equation (4). There is an infinity of distinct paths between a chosen origin and any other bond. New ideas are required to solve such a model.

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